ELSEVIER Journal of Geometry and Physics 20 (1996) 233–249

JOURNAL OF GEOMETRY AND PHYSICS

Decomposition of derivatives and compatibility conditions

Gianluca Gemelli [1]

Scuola di dottorato di ricerca in matematica, Università degli studi di Roma "La Sapienza", Piazzale A.Moro, 00185 Roma, Italy

Received 21 March 1995

Abstract

Some formulae for decomposition of partial and covariant derivatives of any order are established. From these we obtain the compatibility conditions for the jump of derivatives of scalar functions and tensors which are regularly discontinuous across a hypersurface. Thus the formulations of ordinary compatibility conditions and of "distributional" compatibility conditions are re-expressed in a noniterative form, extended to any order, and related to each other.

Subj. Class.: Classical field theory
1991 MSC: 53B50
Keywords: Regularly discontinuous functions; Compatibility conditions

1. Introduction

In space–time *compatibility conditions* relate the jump of partial derivatives of a regularly discontinuous function (a physical variable), while crossing a hypersurface Σ (*wave front* of a *discontinuity wave*), to the normal vector and its derivatives, by means of some functions which are characteristic of the discontinuity. They are essential for the study of wave propagation phenomena in relativity [8,15–17,20,23], and it can be shown [1,2] that they unify, as their space and time components, the well-known *geometric* and *kinematic* compatibility conditions of classical mechanics [3,4,7,13,22,24].

A general compact but implicit formula, for any order of derivation, of these conditions was given by Cattaneo [5,6] (see Section 4.6).

[1] Correspondence address: Via al Quarto Miglio, 24 00178 Roma, Italy. E-mail: gemelli@ mercurio. mat. uniroma1.it.

SSDI 0393-0440(95)00058-5

Similar conditions hold for tensors; these conditions are usually formulated [2,18] up to the second order, and can be extended to higher orders by iterative derivation. Another method, due to Lichnerowicz [15–17], gives the same results in terms of distributions and tensor distributions, also in this case up to the second order of derivation, and suitable of an iterative extension [9].

Here we establish, in the local geometry framework, a decomposition formula for iterated partial derivatives of any order (7) and one for covariant derivatives (38). From the application of these formulae we obtain the general compatibility conditions (47) and (49) in a direct (not iterative) form. Some other decomposition formulae (9), (41) are given, from which the "distributional" compatibility conditions (52) are obtained, also in this case in direct form and for any order of derivation.

The introduction of ordinary and distributional *infinitesimal discontinuity* (46) and (51) clarifies the relation between the two formulations of the compatibility conditions, and allows to move easily from one to the other.

2. Decomposition of iterated partial derivatives

2.1. *Inner and exterior derivatives*

Let V_{N+1} be a differentiable manifold, x its generic point. Greek indices run from 0 to N and latin ones from 1 to N except where otherwise stated. In $\Omega \subset V_{N+1}$, open and connected set with compact closure, let $f \in C^m(\Omega)$, $m \in \mathbb{N}$, $m \geq 1$. Let $\ell_\alpha \overset{\text{def}}{=} \partial_\alpha f$ and $\ell_0 \neq 0$ in Ω. Following Cattaneo [5,6], we introduce the *inner* (to the generic hypersurface Σ: $f =$ const.) *partial derivatives* $\tilde{\partial}_\alpha$ and the *exterior* partial derivative $\breve{\partial}$:

$$\tilde{\partial}_\alpha \overset{\text{def}}{=} \partial_\alpha - \frac{\ell_\alpha}{\ell_0}\partial_0, \qquad \breve{\partial} \overset{\text{def}}{=} \frac{1}{\ell_0}\partial_0. \tag{1}$$

We have the following decomposition of the operator ∂_α:

$$\partial_\alpha = \tilde{\partial}_\alpha + \ell_\alpha \breve{\partial}. \tag{2}$$

These derivatives have the following properties: $\tilde{\partial}_0 \equiv 0$, $[\tilde{\partial}_\alpha, \tilde{\partial}_\beta] = [\breve{\partial}, \tilde{\partial}_\alpha] = 0$. About the iterated partial derivatives of orders 2 and 3, we have, from (2),

$$\partial_\beta \partial_\alpha = \tilde{\partial}_\beta \tilde{\partial}_\alpha + \ell_{\alpha\beta}\breve{\partial} + \ell_\beta \tilde{\partial}_\alpha \breve{\partial} + \ell_\alpha \tilde{\partial}_\beta \breve{\partial} + \ell_\beta \ell_\alpha \breve{\partial}^2, \tag{3}$$

$$\begin{aligned}\partial_\rho \partial_\beta \partial_\alpha = {}& \tilde{\partial}_\rho \tilde{\partial}_\beta \tilde{\partial}_\alpha + \ell_{\alpha\beta\rho}\breve{\partial} + \ell_{\beta\rho}\tilde{\partial}_\alpha \breve{\partial} + \ell_{\alpha\rho}\tilde{\partial}_\beta \breve{\partial} + \ell_{\alpha\beta}\tilde{\partial}_\rho \breve{\partial} \\ & + \ell_\rho \tilde{\partial}_\beta \tilde{\partial}_\alpha \breve{\partial} + \ell_\beta \tilde{\partial}_\rho \tilde{\partial}_\alpha \breve{\partial} + \ell_\alpha \tilde{\partial}_\rho \tilde{\partial}_\beta \breve{\partial} + \ell_{\alpha\rho}\ell_\beta \breve{\partial}^2 + \ell_{\beta\rho}\ell_\alpha \breve{\partial}^2 \\ & + \ell_{\alpha\beta}\ell_\rho \breve{\partial}^2 + \ell_\rho \ell_\beta \tilde{\partial}_\alpha \breve{\partial}^2 + \ell_\rho \ell_\alpha \tilde{\partial}_\beta \breve{\partial}^2 + \ell_\beta \ell_\alpha \tilde{\partial}_\rho \breve{\partial}^2 + \ell_\rho \ell_\beta \ell_\alpha \breve{\partial}^3,\end{aligned} \tag{4}$$

where $\ell_{\alpha\beta} \overset{\text{def}}{=} \partial_\beta \ell_\alpha$, $\ell_{\alpha\beta\rho} \overset{\text{def}}{=} \partial_\rho \ell_{\alpha\beta}, \ldots$

In (3) and (4) we recognize a formal polynomial in $\breve{\partial}$ with sums of operators of the kind "$\ell\tilde{\partial}$" as coefficients. With the generic operator of the kind "$\ell\tilde{\partial}$" we mean any operator given by products of components of ℓ and of its derivatives times some operators of inner derivative (example: $\ell_\alpha \ell_{\beta\rho} \tilde{\partial}_\sigma \tilde{\partial}_\nu$), including the cases of a simple "ℓ" product operator (example: $\ell_\alpha \ell_\beta \ell_{\rho\sigma}$), and that of a simple "$\tilde{\partial}$" inner derivation operator (example: $\tilde{\partial}_\rho \tilde{\partial}_\beta \tilde{\partial}_\alpha$).

In order to recognize the same polynomial structure in any order of derivation, which, as will be shown in Section 4, has interesting consequences for the compatibility conditions, we are soon going to define the suitable general operator coefficients.

2.2. *Multi-indices*

We indicate a multi-index with a capital latin letter.

If A is a multi-index, let $|A|$ be the number of indices of A: for example, if $A = \alpha_n \alpha_{n-1} \dots \alpha_1$, $|A| = n$.

If B and C are two multi-indices, let $B + C$ be the multi-index given by their joining: for example, if $B = \beta_m \beta_{m-1} \dots \beta_1$ and $C = \gamma_s \gamma_{s-1} \dots \gamma_1$, then $B + C = \beta_m \dots \beta_1 \gamma_s \dots \gamma_1$. Thus $|B + C| = |B| + |C|$.

If we consider a multi-index B constructed with a subset of a given multi-index A, we write $B \subset A$ (for example, if $A = \alpha\beta\rho\sigma\mu\nu$ and $B = \nu\beta\sigma$, then $B \subset A$).

We say that two multi-indices $B, B' \subset A$ are *different in indices* if they are not constructed with the same subset of indices of A, i.e. if there is no permutation of B that gives B'. Of course, if $|B| \neq |B'|$ then B and B' are also different in indices.

When a multi-index $A = \alpha_n \dots \alpha_1$ is given, we say it defines a *reference decreasing order*. If $B \subset A$ is increasing with respect to this reference decreasing order (for example, if $A = \alpha_6 \alpha_5 \dots \alpha_1$, then $B = \alpha_2 \alpha_3 \alpha_5$ is increasing) we indicate it by the expression $B = B^+$, $B = B^-$ if it is decreasing.

Let the expression $\{B + C = A\}$ indicate the set of all possible different in indices couples (B, C) such that $B, C \subset A$ and $B + C$ is equal to A *or to some permutation of* A.

If $\{B_i\}$ is a finite family of multi-indices with $i = 1, \dots, k$, let $|\{B_i\}| = k$ denote the number of multi-indices of that family.

If A is the multi-index $A = \alpha_n \alpha_{n-1} \dots \alpha_1$, let $\partial_A \stackrel{\text{def}}{=} \partial_{\alpha_n} \partial_{\alpha_{n-1}} \dots \partial_{\alpha_1}$; similarly for other differential operators.

2.3. *The operator coefficients*

Now we define the operator coefficient: if A is a multi-index and $j \in \mathbb{N}$, let

$$P_A^j(\ell, \tilde{\partial}) \stackrel{\text{def}}{=} \sum_{\substack{\{B+C=A\} \\ C=C^-}} \left(\sum_{\substack{\{\Sigma_i B_i = B\} \\ |\{B_i\}|=j,\ B_i=B_i^+}} \prod_i \ell_{B_i} \right) \tilde{\partial}_C. \tag{5}$$

As explained in Section 2.2, the first sum in (5) is over all possible couples (B, C), $B, C \subset A$, different in indices and with C decreasing, such that $B + C$ is equal to A or some permutation of A; the second sum is over all possible different sets $\{B_i\}$ of j increasing multi-indices such that $\sum_i B_i$ is equal to B or to some permutation of B.

In other words (5) is the operator given by the sum of all possible operators of the kind "$\ell\tilde{\partial}$" such that:

(1) are constructed with all the indices $\alpha_n \ldots \alpha_1$ of A;
(2) contain j factors of the kind "ℓ";
(3) indices are taken in decreasing order in the terms of the kind $\tilde{\partial}_C$, and in increasing order in the factors ℓ_{B_i} (with respect to the reference decreasing order of $A = \alpha_n \ldots \alpha_1$), as shown by $-$ and $+$ symbols in (5).

It is easy to show that (2) is equivalent to

(2′) contain $|A| - j$ derivatives, counted among those in $\tilde{\partial}_C$ and those of the components of the gradient vectors ℓ_α.

As for example (we omit for brevity the "$(\ell, \tilde{\partial})$"):

$$
\begin{aligned}
P^0_A &= \tilde{\partial}_A \sim P^n_A = \ell_{\alpha_n}\ell_{\alpha_{n-1}} \cdots \ell_{\alpha_1} \sim P^1_{\beta\alpha} = \ell_\alpha\tilde{\partial}_\beta + \ell_\beta\tilde{\partial}_\alpha + \ell_{\alpha\beta},\\
P^1_{\rho\beta\alpha} &= \ell_\alpha\tilde{\partial}_{\rho\beta} + \ell_\beta\tilde{\partial}_{\rho\alpha} + \ell_\rho\tilde{\partial}_{\beta\alpha} + \ell_{\alpha\beta}\tilde{\partial}_\rho + \ell_{\alpha\rho}\tilde{\partial}_\beta + \ell_{\beta\rho}\tilde{\partial}_\alpha + \ell_{\alpha\beta\rho},\\
P^2_{\rho\beta\alpha} &= \ell_\alpha\ell_\beta\tilde{\partial}_\rho + \ell_\beta\ell_\rho\tilde{\partial}_\alpha + \ell_\alpha\ell_\rho\tilde{\partial}_\beta + \ell_{\alpha\beta}\ell_\rho + \ell_{\alpha\rho}\ell_\beta + \ell_{\beta\rho}\ell_\alpha.
\end{aligned}
$$

Furthermore, let $L^j_A(\ell, \tilde{\partial})$ be the sum of all the terms of $P^j_A(\ell, \tilde{\partial})$ containing only factors of the kind "ℓ", and let $(L\tilde{\partial})^j_A(\ell, \tilde{\partial})$ be the sum of the remaining terms: $(L\tilde{\partial})^j_A \stackrel{\text{def}}{=} P^j_A - L^j_A$. For example we have:

$$
\begin{aligned}
L^0_A &= 0 \sim L^n_A = P^n_A \sim L^1_A = \ell_{\alpha_1\ldots\alpha_n},\\
L^2_{\rho\beta\alpha} &= \ell_{\alpha\beta}\ell_\rho + \ell_{\alpha\rho}\ell_\beta + \ell_{\beta\rho}\ell_\alpha.
\end{aligned}
$$

2.4. Decomposition theorem for partial derivatives

Given two general operators over tensors, $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{AB}$ denote the product operator, which operates on the argument T as follows: $\mathcal{AB}(T) \equiv \mathcal{A}(\mathcal{B}(T))$, and let $(\mathcal{AB})$ denote the one that let $\mathcal{A}$ operate, in a formal way, only on $\mathcal{B}$, and then operates on T: $(\mathcal{AB})(T) \equiv \mathcal{A}(\mathcal{B})(T)$. For example:

$$
\begin{aligned}
\partial_\rho P^1_{\beta\alpha} &= \ell_{\alpha\rho}\tilde{\partial}_\beta + \ell_\alpha\partial_\rho\tilde{\partial}_\beta + \ell_{\beta\rho}\tilde{\partial}_\alpha + \ell_\beta\partial_\rho\tilde{\partial}_\alpha + \ell_{\alpha\beta\rho} + \ell_{\alpha\beta}\partial_\rho,\\
(\partial_\rho P^1_{\beta\alpha}) &= \ell_{\alpha\rho}\tilde{\partial}_\beta + \ell_\alpha\partial_\rho\tilde{\partial}_\beta + \ell_{\beta\rho}\tilde{\partial}_\alpha + \ell_\beta\partial_\rho\tilde{\partial}_\alpha + \ell_{\alpha\beta\rho}.
\end{aligned}
$$

We notice that the operators above differ for the term $\ell_{\alpha\beta}\partial_\rho$, i.e. $L^1_{\beta\alpha}\partial_\rho$.

In the general case, with our notations, it is not difficult to realize that we have the following formulae for the operator coefficients:

$$
\begin{aligned}
(\partial_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1}) &= (\delta_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1}) + \ell_{\alpha_n}(L\tilde{\partial})^j_{\alpha_{n-1}\ldots\alpha_1}\breve{\partial},\\
\partial_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1} &= (\partial_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1}) + L^j_{\alpha_{n-1}\ldots\alpha_1}\partial_{\alpha_n},\\
P^j_{\alpha_n\ldots\alpha_1} &= (\delta_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1}) + L^j_{\alpha_{n-1}\ldots\alpha_1}\tilde{\partial}_{\alpha_n} + \ell_{\alpha_n} P^{j-1}_{\alpha_{n-1}\ldots\alpha_1},
\end{aligned}
\tag{6}
$$

where δ_α is Cattaneo's double-valued differential operator [5,6]:

$$\delta_\alpha \stackrel{\text{def}}{=} \begin{cases} \partial_\alpha & \text{if operating over a factor of the kind "}\ell\text{"}, \\ \tilde{\partial}_\alpha & \text{if operating over a factor of the kind "}\tilde{\partial}\text{"}. \end{cases}$$

Thus we can prove the following theorem.

Theorem 1. *If A is a multi-index, then*

$$\partial_A = \sum_{j=0}^{|A|} P_A^j(\ell, \tilde{\partial})\breve{\partial}^j. \tag{7}$$

Proof. Let $A = \alpha_n \ldots \alpha_1$. After realizing that (2)–(4) give (7) for $n = |A| = 1, 2, 3$, in order to prove (7) by induction, we suppose

$$\partial_{\alpha_{n-1}\ldots\alpha_1} = \sum_{j=0}^{n-1} P^j_{\alpha_{n-1}\ldots\alpha_1}\breve{\partial}^j. \tag{8}$$

By derivation of (8) for ∂_{α_n}, (6) easily lead us to (7). □

2.5. "Scalar × derivatives" formula

Theorem 2. *Let ϕ and ψ be two functions in $C^n(\Omega)$, $n \in \mathbb{N}$, and A be a multi-index with $|A| = n$. Then*

$$\psi\partial_A\phi \cong_\psi \sum_{j=0}^{|A|} P_A^j(\ell, \partial)(-1)^j\breve{\partial}^j\psi\ \phi, \tag{9}$$

where the equivalence $\cong_\psi$ means equality but for terms containing inner derivatives of ψ, and where of course the $P_A^j(\ell, \partial)$ operator coefficients are constructed with the same method (5) of $P_A^j(\ell, \tilde{\partial})$, but with the partial derivatives ∂_α in place of the inner derivatives $\tilde{\partial}_\alpha$.

Proof. For the operators $P_A^j(\ell, \partial)$, instead of (6), the following analogous formulae hold (this time we omit the "(ℓ, ∂)"):

$$\begin{aligned} (\partial_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1}) &= (\delta_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1}), \\ \partial_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1} &= (\partial_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1}) + L^i_{\alpha_{n-1}\ldots\alpha_1}\partial_{\alpha_n}, \\ P^j_{\alpha_n\ldots\alpha_1} &= (\delta_{\alpha_n} P^j_{\alpha_{n-1}\ldots\alpha_1}) + L^i_{\alpha_{n-1}\ldots\alpha_1}\partial_{\alpha_n} + \ell_{\alpha_n} P^{j-1}_{\alpha_{n-1}\ldots\alpha_1}. \end{aligned} \tag{10}$$

From (2) $\partial_\alpha(\phi\psi) = \psi\partial_\alpha\phi + \phi(\tilde{\partial}_\alpha\psi + \ell_\alpha\breve{\partial}\psi) \cong_\psi \psi\partial_\alpha\phi + \ell_\alpha\phi\breve{\partial}\psi$, so (9) is verified for $n = |A| = 1$. Then, in order to prove (9) by induction, we suppose

$$\psi\partial_{\alpha_{n-1}\ldots\alpha_1}\phi \cong_\psi \sum_{j=0}^{n-1} P^j_{\alpha_{n-1}\ldots\alpha_1}(-1)^j\breve{\partial}^j\psi\ \phi, \tag{11a}$$

$$\breve{\partial}\psi\partial_{\alpha_{n-1}\ldots\alpha_1}\phi \cong_\psi \sum_{j=0}^{n-1} P^j_{\alpha_{n-1}\ldots\alpha_1}(-1)^j\breve{\partial}^{j+1}\psi\ \phi. \tag{11b}$$

After derivation of (11a) for ∂_{α_n} and substitution of (11b), formulae (10) easily lead us to (9). □

We furthermore notice that (10) hold also if ϕ or ψ is a distribution over the space of C^n test functions with compact support $K \subset \Omega$, once the definition of product of a function times a distribution and that of derivative of a distribution (see for example [12,14]) are used.

3. Decomposition of iterated covariant derivatives

3.1. Inner and exterior covariant derivatives

Let V_{N+1} be a Riemann manifold, ∇_α its covariant derivative, and $R_{\alpha\beta\rho}{}^{\sigma} = \partial_\beta \Gamma_{\alpha\rho}{}^{\sigma} - \partial_\alpha \Gamma_{\beta\rho}{}^{\sigma} + \Gamma_{\alpha\rho}{}^{\nu}\Gamma_{\beta\nu}{}^{\sigma} - \Gamma_{\beta\rho}{}^{\nu}\Gamma_{\alpha\nu}{}^{\sigma}$ its curvature tensor, where $\Gamma_{\alpha\beta}{}^{\sigma}$ are Christoffel's symbols (see for example [19, p.219] or [11, p.31]).

In $\Omega \subset V_{N+1}$, open and connected set with compact closure, let $f \in C^m(\Omega)$, $m \geq 1$. Let $\ell_\alpha \stackrel{\text{def}}{=} \partial_\alpha f$ and $\ell_0 \neq 0$ in Ω. We define, similar to Section 2.1, the *inner "covariant" derivative* $\tilde{\nabla}_\alpha$ and the *exterior* one $\check{\nabla}$:

$$\tilde{\nabla}_\alpha \stackrel{\text{def}}{=} \nabla_\alpha - \frac{\ell_\alpha}{\ell_0}\nabla_0, \qquad \check{\nabla} \stackrel{\text{def}}{=} \frac{1}{\ell_0}\nabla_0. \tag{12}$$

The result of the operation $\tilde{\nabla}_\alpha$ is still a tensor with respect to coordinate transformations of the type: $x^{i'} = x^{i'}(x^k)$, $x^{0'} = x^{0'}(x^\alpha)$, $i, k = 1, \ldots, N$, $\alpha = 0, \ldots, N$; such transformations also keep $\check{\nabla} = \check{\nabla}'$ unchanged.

Like in Section 2.1 there is the following decomposition:

$$\nabla_\alpha = \tilde{\nabla}_\alpha + \ell_\alpha \check{\nabla}. \tag{13}$$

Again $\tilde{\nabla}_0 \equiv 0$, but this time these operators do not commute, so much more complications arise when we are looking for formulae similar to (7) and (9) but involving covariant instead of partial derivatives. In order to reach this aim, which later will reveal its utility, we are led to introduce, in the following, some suitable differential operators.

It results, from the definitions,

$$\check{\nabla}\tilde{\nabla}_\alpha - \tilde{\nabla}_\alpha\check{\nabla} = \frac{1}{\ell_0}(\nabla_0\nabla_\alpha - \nabla_\alpha\nabla_0), \tag{14}$$

where in the general case *Ricci's formula* (see for example [21, p.140] or [10, p.188]) holds:

$$(\nabla_\beta\nabla_\alpha - \nabla_\alpha\nabla_\beta)T^{\nu\ldots}_{\mu\ldots} = R_{\alpha\beta\sigma}{}^{\nu}T^{\sigma\ldots}_{\mu\ldots} + \cdots - R_{\alpha\beta\mu}{}^{\sigma}T^{\nu\ldots}_{\sigma\ldots} - \cdots. \tag{15}$$

Let us define the operator $\tilde{r}_\alpha \stackrel{\text{def}}{=} (\check{\nabla}\tilde{\nabla}_\alpha - \tilde{\nabla}_\alpha\check{\nabla})$, i.e.,

$$\tilde{r}_\alpha T^{\nu\ldots}_{\mu\ldots} \stackrel{\text{def}}{=} \frac{1}{\ell_0}R_{\alpha 0\sigma}{}^{\nu}T^{\sigma\ldots}_{\mu\ldots} + \cdots - \frac{1}{\ell_0}R_{\alpha 0\mu}{}^{\sigma}T^{\nu\ldots}_{\sigma\ldots} - \cdots. \tag{16}$$

Furthermore we give the iterative definition of a more general operator $\tilde{r}_A$:

$$\tilde{r}_{\alpha_1\ldots\alpha_n} \stackrel{\text{def}}{=} \tilde{\nabla}_{\alpha_n}\tilde{r}_{\alpha_1\ldots\alpha_{n-1}} - \tilde{r}_{\alpha_1\ldots\alpha_{n-1}}\tilde{\nabla}_{\alpha_n}. \tag{17}$$

3.2. Commutation and inversion formulae

By definition (16) of $\tilde{r}_\alpha$, Ricci's formula for the commutation of inner and exterior covariant derivatives is

$$\check{\nabla}\tilde{\nabla}_\beta - \tilde{\nabla}_\beta\check{\nabla} = \tilde{r}_\beta. \tag{18}$$

Now we introduce the operator $\mathcal{R}_A$:

$$\mathcal{R}_A \overset{\text{def}}{=} P_A^1(\tilde{r}, \tilde{\nabla}) \tag{19}$$

(where $P_A^1(\tilde{r}, \tilde{\nabla})$ is obviously the analogue of operator (5) for $j = 1$ with $\tilde{r}$ in place of ℓ and $\tilde{\nabla}$ in place of $\tilde{\partial}$). Since $\mathcal{R}_\alpha \equiv \tilde{r}_\alpha$, the second covariant derivatives can be decomposed as follows:

$$\nabla_\beta\nabla_\alpha = \tilde{\nabla}_\beta\tilde{\nabla}_\alpha + (L_{\alpha\beta} + \ell_\beta\tilde{\nabla}_\alpha + \ell_\alpha\tilde{\nabla}_\beta)\check{\nabla} + \ell_\alpha\ell_\beta\check{\nabla}^2 + \ell_\beta\mathcal{R}_\alpha, \tag{20}$$

where $L_{\alpha\beta} \overset{\text{def}}{=} \nabla_\beta\ell_\alpha$. We have the following:

Theorem 3. *For every multi-index A,*

$$\check{\nabla}\tilde{\nabla}_A - \tilde{\nabla}_A\check{\nabla} = \mathcal{R}_A. \tag{21}$$

Proof. In this case, the formulae analogous to (6) are the following:

$$\begin{aligned} \tilde{\nabla}_{\alpha_n}\mathcal{R}_{\alpha_{n-1}\ldots\alpha_1} &= (\tilde{\nabla}_{\alpha_n}\mathcal{R}_{\alpha_{n-1}\ldots\alpha_1}) + \tilde{r}_{\alpha_{n-1}\ldots\alpha_1}\tilde{\nabla}_{\alpha_n}, \\ \mathcal{R}_{\alpha_n\ldots\alpha_1} &= \tilde{\nabla}_{\alpha_n}\mathcal{R}_{\alpha_{n-1}\ldots\alpha_1} + \tilde{r}_{\alpha_n}\tilde{\nabla}_{\alpha_{n-1}\ldots\alpha_1}. \end{aligned} \tag{22}$$

The validity of (21) for $n = |A| = 1$ is assured by (18). If we suppose

$$\check{\nabla}\tilde{\nabla}_{\alpha_{n-1}\ldots\alpha_1} = \tilde{\nabla}_{\alpha_{n-1}\ldots\alpha_1}\check{\nabla} + \mathcal{R}_{\alpha_{n-1}\ldots\alpha_1}, \tag{23}$$

then, for (18) $\check{\nabla}\tilde{\nabla}_{\alpha_n\ldots\alpha_1} = \tilde{\nabla}_{\alpha_n}\check{\nabla}\tilde{\nabla}_{\alpha_{n-1}\ldots\alpha_1} + \tilde{r}_{\alpha_n}\tilde{\nabla}_{\alpha_{n-1}\ldots\alpha_1}$, so from (23) and (22) we get (21). □

Moreover, let us introduce the operators $\mathcal{R}_\alpha^{(i)}$, $i \in \mathbb{N}$, $i \geq 1$:

$$\mathcal{R}_\alpha^{(i)} T^{\nu\ldots}_{\mu\ldots} \overset{\text{def}}{=} \check{\nabla}^{i-1}\left(\frac{1}{\ell_0}R_{\alpha 0\sigma}{}^{\nu}\right) T^{\sigma\ldots}_{\mu\ldots} + \cdots - \check{\nabla}^{i-1}\left(\frac{1}{\ell_0}R_{\alpha 0\mu}{}^{\sigma}\right) T^{\nu\ldots}_{\sigma\ldots} + \cdots \tag{24}$$

with $\mathcal{R}_\alpha^{(1)} \equiv \mathcal{R}_\alpha \equiv \tilde{r}_\alpha$. For them it results

$$\check{\nabla}\mathcal{R}_\alpha^{(i)} = \mathcal{R}_\alpha^{(i+1)} + \mathcal{R}_\alpha^{(i)}\check{\nabla}. \tag{25}$$

At last we define the formal differential operator

$$\tilde{D}_\alpha \overset{\text{def}}{=} \tilde{\nabla}_\alpha + \mathcal{R}_\alpha\check{\nabla}^{-1}, \tag{26}$$

which, of course, makes sense only if applied to the exterior derivative $\breve{\nabla}T$ of a tensor: $\tilde{D}_\alpha\breve{\nabla}T \equiv \tilde{\nabla}_\alpha\breve{\nabla}T + \mathcal{R}_\alpha T$. Now we can rewrite (18) in the following form:

$$\breve{\nabla}\tilde{\nabla}_\alpha = \tilde{D}_\alpha\breve{\nabla} \tag{27}$$

and (21) as follows:

$$\breve{\nabla}\tilde{\nabla}_A = \tilde{D}_A\breve{\nabla} \tag{28}$$

by introducing the inversion operator

$$\tilde{D}_A \stackrel{\text{def}}{=} \tilde{\nabla}_A + \mathcal{R}_A\breve{\nabla}^{-1}, \tag{29}$$

which also makes sense only when applied to the exterior derivative $\breve{\nabla}T$ of a tensor. It is not difficult to prove, for example by iteration of (27), the important property that the inversion operator $\tilde{D}_A$ can be expressed by product of operators $\tilde{D}_{\alpha_i}$ of (26):

$$\tilde{D}_{\alpha_n\ldots\alpha_1} \equiv \tilde{D}_{\alpha_n}\tilde{D}_{\alpha_{n-1}}\ldots\tilde{D}_{\alpha_1}. \tag{30}$$

If we define the operator $\tilde{D}_\alpha^{(m)}$, $m \geq 0$, by

$$\breve{\nabla}^m\tilde{\nabla}_\alpha \equiv \tilde{D}_\alpha^{(m)}\breve{\nabla}^m \sim \tilde{D}_\alpha^{(m)} \stackrel{\text{def}}{=} \breve{\nabla}^m\tilde{\nabla}_\alpha\breve{\nabla}^{-m}, \quad m \in \mathbb{N}, \tag{31}$$

then the *inversion formulae* hold:

$$\breve{\nabla}^i\tilde{D}_\alpha^{(m)} = \tilde{D}_\alpha^{(i+m)}\breve{\nabla}^i, \quad i \in \mathbb{Z}, \ \ i+m \geq 0. \tag{32}$$

These formulae have a precise meaning once we establish which is the form of the operators $\tilde{D}_\alpha^{(k)}$. For example, from (32) one can easily find:

$$\begin{aligned} \tilde{D}_\alpha^{(0)} &= \tilde{\nabla}_\alpha, \\ \tilde{D}_\alpha^{(1)} &\equiv \tilde{D}_\alpha = \tilde{\nabla}_\alpha + \mathcal{R}_\alpha\breve{\nabla}^{-1}, \\ \tilde{D}_\alpha^{(2)} &= \tilde{\nabla}_\alpha + 2\mathcal{R}_\alpha\breve{\nabla}^{-1} + \mathcal{R}_\alpha^{(2)}\breve{\nabla}^{-2}, \\ \tilde{D}_\alpha^{(3)} &= \tilde{\nabla}_\alpha + 3\mathcal{R}_\alpha\breve{\nabla}^{-1} + 3\mathcal{R}_\alpha^{(2)}\breve{\nabla}^{-2} + \mathcal{R}_\alpha^{(3)}\breve{\nabla}^{-3}. \end{aligned} \tag{33}$$

In the general case it is possible to recognize that

$$\tilde{D}_\alpha^{(k)} = \sum_{i=0}^{k}\binom{k}{i}\mathcal{R}_\alpha^{(i)}\breve{\nabla}^{-i} \tag{34}$$

having defined, $\mathcal{R}_\alpha^0 \stackrel{\text{def}}{=} \tilde{\nabla}_\alpha$. Operator (34) is admissible, i.e. it makes sense, only if applied to the kth exterior derivative $\breve{\nabla}^kT$ of a tensor.

Now we can consider operators of the kind:

$$\tilde{D}_{\alpha_n\ldots\alpha_1}^{(k_n\ldots k_1)} \stackrel{\text{def}}{=} \tilde{D}_{\alpha_n}^{(k_n)}\tilde{D}_{\alpha_{n-1}}^{(k_{n-1})}\ldots\tilde{D}_{\alpha_1}^{(k_1)}. \tag{35}$$

Operators of this kind can be explicited from (31) and (32):

$$\tilde{D}^{(k)}_{\beta}\tilde{D}^{(h)}_{\alpha} = \sum_{i=0}^{k}\binom{k}{i}\mathcal{R}^{(i)}_{\beta}\sum_{j=0}^{h-i}\binom{h-i}{j}\mathcal{R}^{(j)}_{\alpha}\check{\nabla}^{-(i+j)};$$

$$\tilde{D}^{(r)}_{\rho}\tilde{D}^{(k)}_{\beta}\tilde{D}^{(h)}_{\alpha} = \sum_{l=0}^{r}\binom{r}{l}\mathcal{R}^{(l)}_{\rho}\sum_{i=0}^{k-l}\binom{k-l}{i}\mathcal{R}^{(i)}_{\beta} \times \sum_{j=0}^{h-(l+i)}\binom{h-(l+i)}{j}\mathcal{R}^{(j)}_{\alpha}\check{\nabla}^{-(l+i+j)}.$$

The operators above are admissible when $h \geq k \geq r$, and when applied to a $\check{\nabla}^{h}$. For example, if $r = h = k = 1$,

$$\tilde{D}_{\beta}\tilde{D}_{\alpha} = \tilde{\nabla}_{\beta}\tilde{\nabla}_{\alpha} + (\tilde{\nabla}_{\beta}\mathcal{R}_{\alpha} + \mathcal{R}_{\beta}\tilde{\nabla}_{\alpha})\check{\nabla}^{-1}, \tag{36}$$
$$\tilde{D}_{\rho}\tilde{D}_{\beta}\tilde{D}_{\alpha} = \tilde{\nabla}_{\rho}\tilde{\nabla}_{\beta}\tilde{\nabla}_{\alpha} + (\tilde{\nabla}_{\rho}\tilde{\nabla}_{\beta}\mathcal{R}_{\alpha} + \tilde{\nabla}_{\rho}\mathcal{R}_{\beta}\mathcal{R}_{\alpha} + \mathcal{R}_{\rho}\tilde{\nabla}_{\beta}\tilde{\nabla}_{\alpha})\check{\nabla}^{-1}.$$

It is also possible to calculate explicitly the general operator of the kind (35); it results

$$\tilde{D}^{(k_n\ldots k_1)}_{\alpha_n\ldots\alpha_1} = \prod_{j=0}^{n-1}\sum_{i_{n-j}=0}^{k_{n-j}-s_{ij}}\binom{k_n - s_{ij}}{i_{n-j}}\mathcal{R}^{(i_{n-j})}_{\alpha_{i_{n-j}}}\check{\nabla}^{\sigma_{in}}, \tag{37}$$

where $\sigma_{in} \stackrel{\text{def}}{=} -\sum_{l=1}^{n} i_l$ and $s_{ij} \stackrel{\text{def}}{=} \sum_{l=0}^{j-1} i_{n-l}$, and one can recognize that *such an operator is admissible only when*

(a) $k_1 \geq k_2 \geq \cdots \geq k_n$ $(k_i \in \mathbb{N})$,

(b) it is applied to the mth exterior covariant derivative of a tensor $\check{\nabla}^{m}T$ with $m \geq k_1$.

3.3. Decomposition theorem for covariant derivatives

Theorem 4. *If A is a multi-index, then*

$$\nabla_A = \sum_{j=0}^{|A|} P^{j}_{A}(L, \tilde{D}^{(\mathrm{R})})\check{\nabla}^{j}, \tag{38}$$

where the operator coefficients $P^{j}_{A}(L, \tilde{D}^{(\mathrm{R})})$ are constructed with the same method (5) *of $P^{j}_{A}(\ell, \tilde{\partial})$, but with L in place of ℓ (where $L_{\alpha\beta} \stackrel{\text{def}}{=} \nabla_{\beta}\ell_{\alpha}$, $L_{\alpha\beta\rho} \stackrel{\text{def}}{=} \nabla_{\rho}L_{\alpha\beta} \ldots$) and $\tilde{D}$ in place of $\tilde{\partial}$, and the upper indices (k) are assigned to each "$\tilde{D}_{\alpha}$" according to the following rule* (R *of $\tilde{D}^{(\mathrm{R})}$ in* (38) *stands for "rule").*

Rule. Each upper index k of $\tilde{D}^{(k)}$ must solve the equation

$$k = j + 1 - i + a, \tag{39}$$

where:

(1) j is the upper index of the P^{j}_{A} in which the considered "$\tilde{D}_{\alpha_i}$" appears,

(2) i is the index of α_i with respect to the given $A = \alpha_n \ldots \alpha_1$,

(3) a is the total number of indices α_l, with $l < i$, that appear in the same "$L\tilde{D}$" term of the considered $\tilde{D}_{\alpha_i}$ as index of derivation (of the kind $\tilde{D}_{\alpha_l}$, or of covariant derivation of the normal vector ℓ_α, like for example in $L_{\alpha_b \ldots \alpha_l \ldots}$).

Proof. With our notation (13) and (20) can be re-expressed as

$$\begin{aligned} \nabla_\alpha &= \tilde{D}^{(0)}_\alpha + \ell_\alpha \breve{\nabla}, \\ \nabla_\beta \nabla_\alpha &= \tilde{D}^{(0)}_\beta \tilde{D}^{(0)}_\alpha + (L_{\alpha\beta} + \ell_\beta \tilde{D}^{(1)}_\alpha + \ell_\alpha \tilde{D}^{(0)}_\beta)\breve{\nabla} + \ell_\alpha \ell_\beta \breve{\nabla}^2. \end{aligned} \tag{40}$$

They satisfy rule (39), their operators are all admissible in the sense of Section 3.2, and moreover they assure the validity of (38) for $n = |A| = 1, 2$. It is easy to see that the admissibility of an operator is preserved by derivation for ∇_α, and the same occurs, of course, for the polynomial character in $\breve{\nabla}$. So, if we prove that rule (39) is also preserved by derivation, the theorem will be proved.

Let us consider, then, the generic "$L\tilde{D}$" term of $P^j_{\alpha_{n-1}\ldots\alpha_1}$, suppose it satisfies rule (39), and let us derive it by $\nabla_{\alpha_n} = \tilde{\nabla}_{\alpha_n} + \ell_{\alpha_n}\breve{\nabla}$:

(a) ∇_{α_n} acts over the "L" factor of "$L\tilde{D}$" making a new term appear in $P^j_{\alpha_n\ldots\alpha_1}$. The upper indices (k) have not been touched, i, j and a are not changed, so the rule is still satisfied.

(b) $\tilde{\nabla}_{\alpha_n}$ acts over the "$\tilde{D}$" factor of "$L\tilde{D}$", making a new term appear in $P^j_{\alpha_n\ldots\alpha_1}$, a term of the kind "$L\nabla_{\alpha_n}\tilde{D}$". Indices in "$\tilde{D}$" are unchanged, and the index 0 for $\tilde{D}_{\alpha_n}$ ($\tilde{\nabla}_{\alpha_n} \equiv \tilde{D}^{(0)}_{\alpha_n}$) satisfies the rule, because $i = n$ and $a = n - 1 - j$ (for, as seen in Section 2.2, in each term of $P^j_{\alpha_{n-1}\ldots\alpha_1}$ there are exactly $n - 1 - j$ derivatives).

(c) $\ell_{\alpha_n}\breve{\nabla}$ acts over the "$\tilde{D}$" factor of "$L\tilde{D}$", making a new term appear in $P^{j+1}_{\alpha_n\ldots\alpha_1}$. The new indices (k) are now increased by 1 (for the inversion formula (32)), but also j is now increased by 1, so the rule is again satisfied. □

For example we consider

$$\begin{aligned} \nabla_\rho \nabla_\beta \nabla_\alpha = {} & \tilde{\nabla}_\rho \tilde{\nabla}_\beta \tilde{\nabla}_\alpha + (L_{\alpha\beta\rho} + L_{\beta\rho}\tilde{D}_\alpha + L_{\alpha\rho}\tilde{\nabla}_\beta + L_{\alpha\beta}\tilde{\nabla}_\rho + \ell_\rho \tilde{D}_\beta \tilde{D}_\alpha \\ & + \ell_\beta \tilde{\nabla}_\rho \tilde{D}_\alpha + \ell_\alpha \tilde{\nabla}_\rho \tilde{\nabla}_\beta)\breve{\nabla} + (L_{\alpha\rho}\ell_\beta + L_{\beta\rho}\ell_\alpha + L_{\alpha\beta}\ell_\rho + \ell_\rho \ell_\beta \tilde{D}^{(2)}_\alpha \\ & + \ell_\rho \ell_\alpha \tilde{D}_\beta + \ell_\beta \ell_\alpha \tilde{\nabla}_\rho)\breve{\nabla}^2 + \ell_\rho \ell_\beta \ell_\alpha \breve{\nabla}^3, \end{aligned}$$

i.e. by (33) and (37):

$$\begin{aligned} \nabla_\rho \nabla_\beta \nabla_\alpha = {} & \tilde{\nabla}_\rho \tilde{\nabla}_\beta \tilde{\nabla}_\alpha + \ell_\rho \tilde{\nabla}_\beta \mathcal{R}_\alpha + \ell_\rho \mathcal{R}_\beta \tilde{\nabla}_\alpha + L_{\beta\rho}\mathcal{R}_\alpha + \ell_\beta \tilde{\nabla}_\rho \mathcal{R}_\alpha + \ell_\beta \ell_\rho \mathcal{R}^{(2)}_\alpha \\ & + (L_{\alpha\beta\rho} + L_{\beta\rho}\tilde{\nabla}_\alpha + L_{\alpha\rho}\tilde{\nabla}_\beta + L_{\alpha\beta}\tilde{\nabla}_\rho + \ell_\rho \tilde{\nabla}_\beta \tilde{\nabla}_\alpha + \ell_\beta \tilde{\nabla}_\rho \tilde{\nabla}_\alpha \\ & + \ell_\alpha \tilde{\nabla}_\rho \tilde{\nabla}_\beta + 2\ell_\beta \ell_\rho \mathcal{R}_\alpha + \ell_\alpha \ell_\rho \mathcal{R}_\beta)\breve{\nabla} + (L_{\alpha\rho}\ell_\beta + L_{\beta\rho}\ell_\alpha + L_{\alpha\beta}\ell_\rho \\ & + \ell_\rho \ell_\beta \tilde{\nabla}_\alpha + \ell_\rho \ell_\alpha \tilde{\nabla}_\beta + \ell_\beta \ell_\alpha \tilde{\nabla}_\rho)\breve{\nabla}^2 + \ell_\rho \ell_\beta \ell_\alpha \breve{\nabla}^3. \end{aligned}$$

3.4. *"Scalar × derivatives" formula in the covariant case*

Let now ψ be a function $\in C^m(\Omega)$, and let T be a tensor of the same continuity. Similar to (9) we have the following:

Theorem 5. *If A is a multi-index, $j \in \mathbb{N}$, then*

$$\psi \nabla_A T \cong_\psi \sum_{j=0}^{|A|} P_A^j(L, \nabla)(-1)^j \breve{\partial}^j \psi \; T, \tag{41}$$

where this time $\cong_\psi$ means equality but for terms containing $\tilde{D}_\alpha^{(k)}$ of ψ and of their exterior derivatives $\breve{\partial}^k \psi$.

Proof. (41) is easily verified for the first integers. Moreover (10) remain unchanged if one substitutes $P(\ell, \partial)$ with $P(L, \nabla)$ and ∂_α with ∇_α, so (41) can be considered proved. □

Analogous to (10), (41) holds also if ψ is a distribution, the product of a distribution times a tensor being a *tensor distribution* (see [14]).

4. Compatibility conditions

4.1. Discontinuity hypersurface

In Ω let Σ be a hypersurface of equation $f(x) = 0$, $f \in C^m(\Omega)$, $m \geq 1$. Let $\ell_\alpha \stackrel{\text{def}}{=} \partial_\alpha f$ and let $\ell_0 \neq 0$ in Ω. Σ divides Ω into Ω^+ and Ω^- in accordance with the sign of f.

If we consider inner and exterior partial or covariant derivatives (1) and (12), we have that *inner derivatives have on Σ values depending only by the restriction on Σ of the function or tensor they are applied to* [2]. In particular, $\varphi|_\Sigma = \text{const.}$, $T|_\Sigma = \text{const.} \Rightarrow \tilde{\partial}_\alpha \varphi = \tilde{\nabla}_\alpha T = 0$ on Σ.

Let φ be a function $\in C^0(\Omega \backslash \Sigma)$:

$$\varphi(x) = \begin{cases} \varphi^+(x) & \text{if } x \in \Omega^+, \\ \varphi^-(x) & \text{if } x \in \Omega^-, \end{cases}$$

with $\varphi^+ \stackrel{\text{def}}{=} \varphi|_{\Omega^+} \in C(\Omega^+)$ and $\varphi^- \stackrel{\text{def}}{=} \varphi|_{\Omega^-} \in C(\Omega^-)$.

If φ^+ and φ^- admit, in each point of Σ, the limits φ_0^+ and φ_0^- when the point x tends to Σ, respectively for $f \to 0^+$ and $f \to 0^-$, then on Σ the *jump* or *discontinuity* is defined:

$$[\varphi] \stackrel{\text{def}}{=} \varphi_0^+ - \varphi_0^-, \tag{42}$$

and Σ is called a *discontinuity hypersurface* for φ.

Algebraic properties of the jump are:

$$[\varphi + \psi] = [\varphi] + [\psi], \qquad [\varphi\psi] = [\varphi]\bar{\psi} + \bar{\varphi}[\psi]$$

being $\bar{\psi} \stackrel{\text{def}}{=} \frac{1}{2}(\psi^+ + \psi^-)$. In particular, the *product with a continuous function* φ ($[\varphi] = 0$, $\bar{\varphi} = \varphi$) *commutes with the jump operation.*

All these can be extended in a natural way to tensors.

4.2. Regularly discontinuous functions and tensors

Let $\psi \in C^m(\Omega \backslash \Sigma)$ (the same continuity supposed for Σ), and let $\psi^{\pm} \overset{\text{def}}{=} \psi|_{\Omega^{\pm}}$ and their derivatives admit a limit over Σ, such that the jumps $[\psi]$, $[\partial_\alpha \psi], \ldots$ are well defined on Σ up to order m.

Furthermore, let ψ^+ and ψ^- be prolongable in an m-regular manner in an open set $U \supset \Sigma$, i.e. let them be the restrictions to U^+ and U^- of two functions $\psi^{\pm} \in C^m(U)$ whose derivatives take on Σ values corresponding to the limits of the derivatives of $\psi^{\pm}$. In such a case we call ψ an *m-regularly discontinuous function* on Σ.

An m-regular function (i.e. of class $C^m(U)$) is also m-regularly discontinuous (with null discontinuities).

If $m = 0$, we simply say that the function is regularly discontinuous. Finally, we say that the function is m-regularly discontinuous *of order r* if $\psi \in C^{r-1}(\Omega)$, but $\psi \notin C^r(\Omega)$.

It is not difficult to show that *a sufficient condition for $\psi^{\pm} \in C^m(\Omega^{\pm})$ to be m-prolongable is that their limits on Σ and those of their derivatives are bounded.*

Once a prolongation $\psi^{\pm}$ is chosen, it is possible to define the jump of ψ and of its derivatives not only on Σ but wherever in U:

$$\begin{aligned} [\psi] &\overset{\text{def}}{=} \psi^+ - \psi^- \in C^m(U), \\ [\partial_\alpha \psi] &\overset{\text{def}}{=} \partial_\alpha \psi^+ - \partial_\alpha \psi^- \in C^{m-1}(U), \ldots \end{aligned}$$

so that in U the operations of jump and of derivative commute:

$$\partial_\alpha[\psi] = [\partial_\alpha \psi], \quad \partial_\alpha \partial_\beta[\psi] = [\partial_\alpha \partial_\beta \psi], \ldots, \tag{43}$$

and the same is for jump and inner or exterior derivation, being $\ell \in C^{m-1}(U)$.

But, on Σ, $[\psi]$ and its derivatives are independent from the choice of the prolongation.

Therefore in the following, when writing the jump of a regularly discontinuous function and of its derivatives, it is intended that they are considered on the discontinuity hypersurface Σ, the only place where they are well defined; but, for what previously seen, we can apply without any problem the differential operators ∂_α, $\tilde{\partial}_i$, $\breve{\partial}$ to them.

The extension to tensors is trivial.

4.3. Infinitesimal discontinuity

We call *infinitesimal* (or weak) *discontinuity of order n* of the m-regularly discontinuous function ψ, the function

$$\partial^n \psi \overset{\text{def}}{=} [\breve{\partial}^n \psi], \quad n = 0, \ldots, m. \tag{44}$$

We borrow name and notation from Lichnerowicz [15–17], who defines it for tensors, as a tensor distribution (see Section 4.5). $\partial^n \psi$ corresponds to the α_n introduced by Cattaneo in [5].

We denote $[\psi] \equiv \partial^0 \psi$ and $\partial \psi \equiv \partial^1 \psi$.

From the definition it follows: $\partial^{n+1} = \breve{\partial}\partial^n$. The operation of infinitesimal discontinuity has properties similar to those of a derivative:

$$\partial(a\psi) = a\partial\psi \quad \forall a \in \mathbb{R}, \qquad \partial(\varphi + \psi) = \partial\varphi + \partial\psi, \tag{45}$$

which justifies the notation. The typical rule of derivation of a product also holds, but only for continuous functions.

A fundamental property is that if $\varphi \in C^k(\Omega)$, then $\partial^i\varphi \equiv 0$ for $i = 0, \ldots, k$.

With complete analogy we define the infinitesimal discontinuity of a regularly discontinuous tensor T:

$$\partial^n T \overset{\text{def}}{=} [\breve{\nabla}^n T]. \tag{46}$$

It is a tensor of the same order of T, defined on Σ.

4.4. Compatibility conditions

Having introduced the infinitesimal discontinuity, from the decomposition formula (7), by simple "multiplication" for $[\varphi]$ (i.e. applying the formula to φ and then calculating the jump), we obtain, as a corollary, the *compatibility conditions* for partial derivatives of a regularly discontinuous function:

$$[\partial_A\varphi] = \sum_{j=0}^{|A|} P_A^j(\ell, \tilde{\partial})\partial^j\varphi \tag{47}$$

analogue of those by Cattaneo [5,6], but in explicit form (provided one knows the operator coefficients from (5)). Moreover in the particular case that φ is $(|A| - 1)$-continuous (i.e. $\partial^j\varphi \equiv 0$ for $j < |A|$) conditions (47) give those in [24, p.498, formula (176.11)].

As for example we have:

$$[\partial_\alpha\varphi] = \tilde{\partial}_\alpha[\varphi] + \ell_\alpha\partial\varphi, \tag{48a}$$

$$[\partial_\beta\partial_\alpha\varphi] = \tilde{\partial}_\beta\tilde{\partial}_\alpha[\varphi] + (\ell_\beta\tilde{\partial}_\alpha + \ell_\alpha\tilde{\partial}_\beta + \ell_{\alpha\beta})\partial\varphi + \ell_\alpha\ell_\beta\partial^2\varphi, \tag{48b}$$

$$\begin{aligned}[\partial_\rho\partial_\beta\partial_\alpha\varphi] = {} & \tilde{\partial}_\rho\tilde{\partial}_\beta\tilde{\partial}_\alpha[\varphi] + (\ell_{\alpha\beta\rho} + \ell_{\beta\rho}\tilde{\partial}_\alpha + \ell_{\alpha\rho}\tilde{\partial}_\beta + \ell_{\alpha\beta}\tilde{\partial}_\rho + \ell_\rho\tilde{\partial}_\beta\tilde{\partial}_\alpha \\ & + \ell_\beta\tilde{\partial}_\rho\tilde{\partial}_\alpha + \ell_\alpha\tilde{\partial}_\rho\tilde{\partial}_\beta)\partial\varphi + (\ell_{\alpha\rho}\ell_\beta + \ell_{\beta\rho}\ell_\alpha + \ell_{\alpha\beta}\ell_\rho + \ell_\rho\ell_\beta\tilde{\partial}_\alpha \\ & + \ell_\rho\ell_\alpha\tilde{\partial}_\beta + \ell_\beta\ell_\alpha\tilde{\partial}_\rho)\partial^2\varphi + \ell_\rho\ell_\beta\ell_\alpha\partial^3\varphi. \end{aligned} \tag{48c}$$

Similarly, the decomposition formula (38), by multiplication for $[T]$, *in the hypothesis that the curvature tensor is at least of class* $C^{|A|-2}$ (such that all its derivatives appearing in (38) have null jumps) gives:

$$[\nabla_A T] = \sum_{j=0}^{|A|} P_A^j(L, \tilde{D}^{(\mathrm{R})})\partial^j T, \tag{49}$$

which expresses the *compatibility conditions for* covariant derivatives of a regularly discontinuous *tensor*. For example:

$$
\begin{aligned}
[\nabla_\alpha T] &= \tilde{D}^{(0)}_\alpha[T] + \ell_\alpha \partial T,\\
[\nabla_\beta \nabla_\alpha T] &= \tilde{D}^{(0)}_\beta \tilde{D}^{(0)}_\alpha[T] + (L_{\alpha\beta} + \ell_\alpha \tilde{D}^{(0)}_\beta + \ell_\beta \tilde{D}^{(1)}_\alpha)\partial T + \ell_\alpha \ell_\beta \partial^2 T,\\
[\nabla_\rho \nabla_\beta \nabla_\alpha T] &= \tilde{\nabla}_\rho \tilde{\nabla}_\beta \tilde{\nabla}_\alpha[T] + (L_{\alpha\beta\rho} + L_{\beta\rho}\tilde{D}_\alpha + L_{\alpha\rho}\tilde{\nabla}_\beta + L_{\alpha\beta}\tilde{\nabla}_\rho + \ell_\rho \tilde{D}_\beta \tilde{D}_\alpha\\
&\quad + \ell_\beta \tilde{\nabla}_\rho \tilde{D}_\alpha + \ell_\alpha \tilde{\nabla}_\rho \tilde{\nabla}_\beta)\partial T + (L_{\alpha\rho}\ell_\beta + L_{\beta\rho}\ell_\alpha + L_{\alpha\beta}\ell_\rho\\
&\quad + \ell_\rho \ell_\beta \tilde{D}^{(2)}_\alpha + \ell_\rho \ell_\alpha \tilde{D}_\beta + \ell_\beta \ell_\alpha \tilde{\nabla}_\rho)\partial^2 T + \ell_\rho \ell_\beta \ell_\alpha \partial^3 T,
\end{aligned}
$$

i.e. by (33) and (37)

$$
\begin{aligned}
[\nabla_\alpha T] &= \tilde{\nabla}_\alpha[T] + \ell_\alpha \partial T,\\
[\nabla_\beta \nabla_\alpha T] &= \tilde{\nabla}_\beta \tilde{\nabla}_\alpha[T] + (L_{\alpha\beta} + \ell_\alpha \tilde{\nabla}_\beta + \ell_\beta \tilde{\nabla}_\alpha)\partial T + \ell_\alpha \ell_\beta \partial^2 T + \ell_\beta \mathcal{R}_\alpha[T],\\
[\nabla_\rho \nabla_\beta \nabla_\alpha T] &= (\tilde{\nabla}_\rho \tilde{\nabla}_\beta \tilde{\nabla}_\alpha + \ell_\rho \tilde{\nabla}_\beta \mathcal{R}_\alpha + \ell_\rho \mathcal{R}_\beta \tilde{\nabla}_\alpha + L_{\beta\rho}\mathcal{R}_\alpha + \ell_\beta \tilde{\nabla}_\rho \mathcal{R}_\alpha\\
&\quad + \ell_\beta \ell_\rho \mathcal{R}^{(2)}_\alpha)[T] + (L_{\alpha\beta\rho} + L_{\beta\rho}\tilde{\nabla}_\alpha + L_{\alpha\rho}\tilde{\nabla}_\beta + L_{\alpha\beta}\tilde{\nabla}_\rho + \ell_\rho \tilde{\nabla}_\beta \tilde{\nabla}_\alpha\\
&\quad + \ell_\beta \tilde{\nabla}_\rho \tilde{\nabla}_\alpha + \ell_\alpha \tilde{\nabla}_\rho \tilde{\nabla}_\beta + 2\ell_\beta \ell_\rho \mathcal{R}_\alpha + \ell_\alpha \ell_\rho \mathcal{R}_\beta)\partial T\\
&\quad + (L_{\alpha\rho}\ell_\beta + L_{\beta\rho}\ell_\alpha + L_{\alpha\beta}\ell_\rho + \ell_\rho \ell_\beta \tilde{\nabla}_\alpha + \ell_\rho \ell_\alpha \tilde{\nabla}_\beta + \ell_\beta \ell_\alpha \tilde{\nabla}_\rho)\partial^2 T\\
&\quad + \ell_\rho \ell_\beta \ell_\alpha \partial^3 T
\end{aligned}
$$

(see [2, p.63]).

It must be said that, even if compact and expressive, (49) contains the unusual operators $\tilde{D}^{(k)}$, which must be explicitated by (34) and (35) (or by means of (37)), like in the examples above, so it is not completely satisfying. Anyway it is direct and not iterative, for once (34) is known, (49) does not contain anymore the $\tilde{r}_A$ or other operators defined by iteration. In the tensor case, anyway, the formulation in terms of tensor distributions is much more useful and expressive, as we are going to see in the following section.

4.5. "Distributional" compatibility conditions

We denote by a boldface letter a distribution or a tensor distribution and refer to [12,14–17] for a general treatment of *tensor distributions* and for the definition of *Dirac's mensure* distribution $\boldsymbol{\delta}$ associated to Σ. We notice that, with our notations, a fundamental property of $\boldsymbol{\delta}$ is $\tilde{\partial}_\alpha \boldsymbol{\delta} \equiv 0$, and thus, more generally:

$$\tilde{\partial}_A \boldsymbol{\delta} \equiv 0 \quad \forall A \ \text{multi-index}. \tag{50}$$

Of course, $\boldsymbol{\delta}$ being a distribution, i.e. a scalar distribution, it also results $\tilde{\nabla}_A \boldsymbol{\delta} = 0$ and $\mathcal{R}_\alpha \boldsymbol{\delta} = 0$, so the $\cong_{\boldsymbol{\delta}}$ relation introduced in Sections 2.4 and 3.4, becomes, in the case of $\boldsymbol{\delta}$, equality.

Now we define the *distributional infinitesimal discontinuity* of a tensor:

$$\boldsymbol{\partial}^m T \stackrel{\text{def}}{=} (-1)^m (\breve{\partial}^m \boldsymbol{\delta})[T]. \tag{51}$$

It could seem strange that definition (51) works also for continuous tensors, but its meaning gets clear once one considers the definition of product of a tensor and a distribution, and that of derivative of a distribution [14–17].

$\boldsymbol{\partial}^m T$ is in the general case a tensor distribution defined on Σ. In the particular case that $T = \varphi$ is a function, it is a distribution.

If $\partial^j T = 0\ \forall j < m$ then the relation between infinitesimal discontinuity and distributional infinitesimal discontinuity is very simple: $\boldsymbol{\partial}^m T = \boldsymbol{\delta}\partial^m T$, otherwise it is given by the definition (51).

Now from the "scalar × derivatives" formulae (9) and (41), we get (by putting $\boldsymbol{\delta}$ in place of ψ, $[\varphi]$ in place of ϕ and $[T]$ in place of T) the *distributional compatibility conditions*, which hold without any hypothesis of continuity for the curvature tensor:

$$\boldsymbol{\delta}[\partial_A \varphi] = \sum_{j=0}^{|A|} P_A^j(\ell, \partial)\boldsymbol{\partial}^j \varphi, \tag{52a}$$

$$\boldsymbol{\delta}[\nabla_A T] = \sum_{j=0}^{|A|} P_A^j(L, \nabla)\boldsymbol{\partial}^j T, \tag{52b}$$

(52b) extends to any order of derivation the analogous formulae by Lichnerowicz [15–17]. Examples of (52b):

$$\begin{aligned} \boldsymbol{\delta}[\nabla_\alpha T] &= \nabla_\alpha(\boldsymbol{\delta}[T]) + \ell_\alpha \boldsymbol{\partial} T, \\ \boldsymbol{\delta}[\nabla_\beta \nabla_\alpha T] &= \nabla_\beta \nabla_\alpha(\boldsymbol{\delta}[T]) + (L_{\alpha\beta} + \ell_\alpha \nabla_\beta + \ell_\beta \nabla_\alpha)\boldsymbol{\partial} T + \ell_\alpha \ell_\beta \boldsymbol{\partial}^2 T, \\ \boldsymbol{\delta}[\nabla_\rho \nabla_\beta \nabla_\alpha T] &= \nabla_\rho \nabla_\beta \nabla_\alpha(\boldsymbol{\delta}[T]) + (L_{\alpha\beta\rho} + L_{\beta\rho}\nabla_\alpha + L_{\alpha\rho}\nabla_\beta + L_{\alpha\beta}\nabla_\rho \\ &\quad + \ell_\rho \nabla_\beta \nabla_\alpha + \ell_\beta \nabla_\rho \nabla_\alpha + \ell_\alpha \nabla_\rho \nabla_\beta)\boldsymbol{\partial} T + (L_{\alpha\rho}\ell_\beta + L_{\beta\rho}\ell_\alpha \\ &\quad + L_{\alpha\beta}\ell_\rho + \ell_\rho \ell_\beta \nabla_\alpha + \ell_\rho \ell_\alpha \nabla_\beta + \ell_\beta \ell_\alpha \nabla_\rho)\boldsymbol{\partial}^2 T + \ell_\rho \ell_\beta \ell_\alpha \boldsymbol{\partial}^3 T. \end{aligned} \tag{53}$$

Lichnerowicz demonstrated the existence of two tensor distributions $\boldsymbol{\partial} T$ and $\boldsymbol{\partial}^2 T$ (the last one he denotes $\mathbf{T}'$ or $\overline{\mathbf{T}}$) such that (52a) and (52b) hold, the second under further hypothesis of a continuous T (i.e. $[T] \equiv 0$), without giving their explicit form.

Formulae (52) are direct, expressive, and do not contain strange operators. It is clear that, once (44) and (51) are known, (47) is equivalent to (52a) and, if the curvature tensor is at least of class $C^{|A|-2}$, (49) is equivalent to (52b), such that the two formulations can be considered "dual".

4.6. *"Implicit" formulations*

With our notations Cattaneo's original formula of the general compatibility conditions for a regularly discontinuous function [5] is:

$$[\partial_A \varphi] = \left\{ \delta_A \left(\sum_{k=0}^{|A|} \frac{1}{k!} \partial^k \varphi f^k \right) \right\}_{f=0}, \tag{54}$$

where δ_α is the double-valued differential operator introduced in Section 2.4 and where f is included, by definition, among the factors of the kind "ℓ", and ∂ among those of the kind "$\tilde{\partial}$". This formula is very expressive and compact, but "implicit", in the sense that it still needs $|A|$ iterated derivations by δ_α (and subsequently the position $f = 0$), thus in the applications it does not give much improvement to the method of simple iterated derivation of (48a). One can obtain a similar formula, involving the general differential operators $\tilde{D}_\alpha^{(i)}$, also for the covariant derivatives of a regularly discontinuous tensor, but it of course gives the same problems mentioned at the end of Section 4.4, so we are not going to consider it.

It is of more interest to apply the same method used in [5] to the case of the distributional compatibility conditions. We find that (52) are also susceptible of the following implicit formulation:

$$\delta[\partial_A\varphi] = \left\{\partial_A\left(\sum_{k=0}^{|A|}\frac{1}{k!}\partial^k\varphi f^k\right)\right\}_{f=0}, \quad \delta[\nabla_A T] = \left\{\nabla_A\left(\sum_{k=0}^{|A|}\frac{1}{k!}\partial^k T f^k\right)\right\}_{f=0}. \tag{55}$$

These implicit formulations (54) and (55) counter-balance in part the above-mentioned difficulties by their independence from the complicate definition (5) of the operator coefficients. Anyway our "explicit" formulations (47)–(52) undoubtedly have the good qualities of clearly displaying the polynomial structure in the infinitesimal discontinuity and of not needing any iteration.

References

[1] A.M. Anile, A geometrical characterization of the compatibility relations for regularly discontinuous tensor fields, Le matematiche 37 (1982) 105–119.

[2] A.M. Anile, *Relativistic Fluids and Magneto-Fluids* (Cambridge Univ. Press, Cambridge, 1989) Ch.3.

[3] A.M. Anile, J.K. Hunter, P. Pantano and G. Russo, *Ray Methods for Nonlinear Waves in Fluids and Plasmas* (Longman, Harlow, 1993).

[4] G. Boillat, *La propagation des ondes* (Gauthier-Villars, Paris, 1965).

[5] C. Cattaneo, Funzioni regolarmente discontinue attraverso una ipersuperficie: formula generale di compatibilità geometrica, Istit. Lombardo Accad. Sci. Lett. Rend. A 112 (1978) 139–149.

[6] C. Cattaneo, *Elementi di teoria della propagazione ondosa,* Quaderni dell'Unione Matematica Italiana, 20 (Pitagora Editrice, Bologna 1981) Ch.II.

[7] P.J. Chen, *Selected Topics in Wave Propagation* (Noordhoff, Leyden, 1976).

[8] D.G.B. Edelen and T.Y. Thomas, The dynamics of discontinuity surfaces in general relativity theory, J. Math. Anal. Appl. 7 (1963) 247–263.

[9] R. Estrada and R.P. Kanwal, Applications of distributional derivatives to wave propagation, J. Inst. Math. Appl. 26 (1980) 39–63.

[10] G. Ferrarese, *Introduzione alla dinamica riemanniana dei sistemi continui* (Pitagora, Bologna, 1979).

[11] G. Ferrarese, *Lezioni di relatività generale* (Pitagora, Bologna, 1994).

[12] G. Friedlander, *The Wave Equation on a Curved Space–Time* (Cambridge Univ. Press, Cambridge, 1975).

[13] J. Hadamard, *Leçons sur la propagation des ondes* (Herman, Paris, 1903).

[14] A. Lichnerowicz, *Propagateurs et commutateurs en relativité générale*, Publ. Math. Inst. Ét. Scient. n.10 (1961).

[15] A. Lichnerowicz, Théorie des rayons en hydrodynamique et magnétohydrodynamique relativistes, Ann. Inst. H. Poincaré. 8 (1967) 271–302.

[16] A. Lichnerowicz, Ondes de choc et hipothèses de compressibilité en magnétohydrodynamique relativiste, Commun. Math. Phys. 9 (1969) 145–174.

[17] A. Lichnerowicz, *Ondes de choc, ondes infinitesimales et rayons en hydrodynamique et magnétohydrodynamique relativistes*, Centr. Int. Mat. Est. 1970, Cremonese, Roma (1971).

[18] G. Maugin, Conditions de compatibilité pour une hypersurface singulière en mécanique relativistes des milieux continus, Ann. Inst. H. Poincaré. 24 (1976) 213–241.

[19] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman, New York, 1973).
[20] S. O'Brien and J.L. Synge, Jump conditions at discontinuity in general relativity, Comm. Dublin Inst. Adv. Stud. Ser. A 9 (1952) 1–20.
[21] J.A. Schouten, *Ricci-calculus* (Springer, Berlin, 1954).
[22] T.Y. Thomas, Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics, J. Math. Mech. 6 (1957) 311–322.
[23] T.Y. Thomas, Hypersurfaces in Einstein–Riemann space and their compatibility conditions, J. Math. Anal. Appl. 7 (1963) 225–246.
[24] C. Truesdell and R.A. Toupin, *The Classical Field Theories*, Handbuch der Physik, Bd. III/1 (Berlin, 1960) pp. 266–793.